Fermi National Accelerator Laboratory

FERMILAB-Pub-98/275

Formation of Patterns and Coherent Structures in Charged Particle Beams

Stephan I. Tzenov

Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

October 1998

Operated by Universities Research Association Inc. under Contract No. DE-AC02-76CH03000 with the United States Department of Energy

Fermilab-Pub-98/275

Formation of Patterns and Coherent Structures in Charged Particle Beams

Stephan I. Tzenov
Fermi National Accelerator Laboratory
P. O. Box 500, Batavia, IL 60510, USA
E. mail: tzenov@fnal.gov

In the present paper we study the long wavelength and slow time scale behavior of a coasting beam in a resonator adopting a broad-band impedance model. Based on the renormalization group approach we derive a set of coupled evolution equations for the beam envelope distribution function and the resonator voltage amplitude. The equation for the resonator voltage amplitude is further transformed into a generalized Ginzburg-Landau equation.

I. INTRODUCTION.

So far nonlinear wave phenomena have received scant attention in the study of collective effects in charged particle beams. Considerable experimental and simulation data however exists suggesting that these phenomena should be included into the entire physical picture of beam propagation in accelerators and storage rings.

A vast literature in the field of plasma physics is dedicated to the study of nonlinear wave-particle processes due to space charge interparticle forces. In high energy particle accelerators, where space charge forces are negligibly small, of particular interest is the coherent state of the beam under the influence of wakefields, or in the frequency domain, machine impedance. This state is highly nonlinear and depends on the interaction of nonlinear waves, involving some weak dissipative mechanisms that balance beam fluctuations driven by the wakefields.

In a previous work [1], [2] we studied nonlinear behavior of a coasting beam under the influence of a resonator impedance. Starting from the gas-dynamic equations for longitudinal motion and using a renormalization group (RG) approach [3], [4] we found a set of coupled nonlinear equations for the beam density and resonator voltage. However, as is well-known, the hydrodynamic approximation is valid when the beam is close to a local equilibrium, which in a number of practically important cases may well be far from reality.

The present paper, providing a complete kinetic description of the processes involved, is aimed to overcome the above mentioned difficulties. In what follows we study the longitudinal dynamics of a coasting beam in a resonator adopting a broad-band impedance model. We are interested in describing slow motion of beam patterns (droplets) neglecting fast oscillations of beam density and voltage on the resonator at a frequency close to the resonant frequency. We employ the RG method to derive amplitude equations governing the dynamics of slow processes. In Section II we obtain the desired equations for the longitudinal envelope distribution function and the amplitude of the resonator voltage. In Section III we proceed to transform the equation for the voltage amplitude into a generalized Ginzburg-Landau equation by solving explicitly the Vlasov equation for the envelope distribution function. Finally in Section IV we draw some conclusions resulting from the work performed.

II. THE AMPLITUDE EQUATIONS.

The starting point for the subsequent analysis is the system of equations:

$$\frac{\partial f}{\partial T} + v\frac{\partial f}{\partial \theta} + \lambda V\frac{\partial f}{\partial v} = 0,$$

$$\frac{\partial^2 V}{\partial T^2} + 2\gamma\frac{\partial V}{\partial T} + \omega^2 V = \frac{\partial I}{\partial T}, \tag{1}$$

$$I(\theta;T) = \int dv v f(\theta, v; T),$$

for the longitudinal distribution function $f(\theta, v; T)$ of an unbunched beam and the variation per turn of the voltage $V(\theta; T)$ on a resonator. All dependent and independent variables, as well as free parameters in equations (1) are dimensionless and have been rescaled according to the relations:

$$T = \omega_s t \quad ; \quad v = \frac{1}{\omega_s}\frac{d\theta}{dt} = 1 + \frac{k_o \Delta E}{\omega_s} \quad ; \quad \omega = \frac{\omega_R}{\omega_s} \quad ; \quad \gamma = \frac{\omega}{2Q}, \tag{2}$$

$$\lambda = \frac{e^2 R \gamma k_o \rho_o}{\pi}. \tag{3}$$

Here ω_s is the angular revolution frequency of the synchronous particle, ΔE is the energy error, ω_R is the resonant frequency, Q is the quality factor of the resonator, R is the resonator shunt impedance and ρ_o is the uniform beam density distribution at the thermodynamic limit. Furthermore

$$k_o = -\frac{\eta \omega_s}{\beta_s^2 E_s} \tag{4}$$

is the proportionality constant between the frequency deviation and the energy deviation of a non synchronous particle with respect to the synchronous one, while $\eta = \alpha_M - \gamma_s^{-2}$ (α_M - momentum compaction factor) is the phase slip coefficient. The voltage variation per turn V, the beam current I and the longitudinal distribution function f entering equations (1) have been rescaled as well from their actual values V_a, I_a and f_a as follows:

$$V_a = 2e\omega_s \rho_o \gamma R V \quad ; \quad I_a = e\omega_s \rho_o I \quad ; \quad f_a = \rho_o f \tag{5}$$

Let us introduce the Radon transform [5], [6] of the distribution function $f(\theta, v; T)$

$$f(\theta, v; T) = \int d\xi F(\theta, \xi; T)\, \delta[v - U(\theta, \xi; T)]. \tag{6}$$

In the definition (6) ξ can be viewed as a Lagrange variable which is usually determined from the condition that the distribution function $f(\theta, v; T)$ be equal to a specified distribution, say the equilibrium distribution for instance:

$$f(\theta, v; T) = f_0(\xi) \qquad \Rightarrow \qquad v = U(\theta, \xi; T).$$

Substitution of eq. (6), into the system (1) yields:

$$\frac{\partial F}{\partial T} + \varepsilon \frac{\partial}{\partial \theta}(FU) = 0,$$

$$\frac{\partial U}{\partial T} + \varepsilon U \frac{\partial U}{\partial \theta} = \lambda V, \tag{7}$$

$$\frac{\partial^2 V}{\partial T^2} + 2\gamma \frac{\partial V}{\partial T} + \omega^2 V = \int d\xi \frac{\partial}{\partial T}(FU),$$

where the fact that the azimuth θ is a slow variable (the dependence of F, U and V on θ is through a stretched variable $\zeta = \varepsilon\theta$) has been taken into account. Note that the system (7) resembles the set of gas-dynamic equations, governing the longitudinal motion of the beam. It bears however, additional information about the velocity distribution, embedded in the dependence on the Lagrange variable ξ, and takes into account its overall effect through the integral on the right hand side of the third equation.

We next examine the solution of the system of equations (7) order by order in the formal small parameter ε by carrying out a naive perturbation expansion. The zero order solution (stationary solution) is readily found to be:

$$F_0 = F_0(\xi) \quad ; \quad U_0 = U_0(\xi) \quad ; \quad V_0 \equiv 0$$

and in particular one can choose

$$F_0(\xi) = f_0(\xi) \qquad ; \qquad U_0(\xi) = 1 + \xi.$$

Combining the first order equations

$$\frac{\partial F_1}{\partial T} = 0 \qquad ; \qquad \frac{\partial U_1}{\partial T} = \lambda V_1,$$

$$\frac{\partial^2 V_1}{\partial T^2} + 2\gamma \frac{\partial V_1}{\partial T} + \omega^2 V_1 = \int d\xi F_0 \frac{\partial U_1}{\partial T},$$

yields trivially a unique equation for V_1:

$$\frac{\partial^2 V_1}{\partial T^2} + 2\gamma \frac{\partial V_1}{\partial T} + \omega_o^2 V_1 = 0 \qquad ; \qquad \omega_o^2 = \omega^2 - \lambda.$$

Solving the first order equations one easily obtains:

$$V_1 (\theta; T) = E (\theta; T_0) e^{i\omega_1 \Delta T} + c.c.$$

$$U_1 (\theta, \xi; T) = u_o (\theta, \xi; T_0) + \lambda \frac{E (\theta; T_0)}{i\omega_1} e^{i\omega_1 \Delta T} + c.c. \tag{8}$$

$$F_1 (\theta, \xi; T) = R_o (\theta, \xi; T_0),$$

where

$$\omega_1 = \omega_q + i\gamma \quad ; \quad \omega_q^2 = \omega_o^2 - \gamma^2 \quad ; \quad \Delta T = T - T_0, \tag{9}$$

and the amplitudes $E (\theta; T_0)$, $u_o (\theta, \xi; T_0)$, $R_o (\theta, \xi; T_0)$ are yet unknown functions of θ, ξ and the initial instant of time T_0. Proceeding further with the expansion in the formal parameter ε we write down the second order equations

$$\frac{\partial F_2}{\partial T} + F_0 \frac{\partial U_1}{\partial \theta} + U_0 \frac{\partial R_o}{\partial \theta} = 0,$$

$$\frac{\partial U_2}{\partial T} + U_0 \frac{\partial U_1}{\partial \theta} = \lambda V_2,$$

$$\frac{\partial^2 V_2}{\partial T^2} + 2\gamma \frac{\partial V_2}{\partial T} + \omega^2 V_2 = \int d\xi \left(F_0 \frac{\partial U_2}{\partial T} + \lambda R_o V_1 + U_0 \frac{\partial F_2}{\partial T} \right)$$

and by elimination of U_2 and F_2 from the third equation we obtain:

$$\frac{\partial^2 V_2}{\partial T^2} + 2\gamma \frac{\partial V_2}{\partial T} + \omega_o^2 V_2 = \lambda V_1 \int d\xi R_o - 2 \int d\xi U_0 F_0 \frac{\partial U_1}{\partial \theta} - \int d\xi U_0^2 \frac{\partial R_o}{\partial \theta}.$$

Solving the above equation and subsequently the two other equations for U_2 and F_2 we find the second order solution as follows:

$$V_2 (\theta; T) = -\frac{1}{\omega_o^2} \int d\xi \left(U_0^2 \frac{\partial R_o}{\partial \theta} + 2F_0 U_0 \frac{\partial u_o}{\partial \theta} \right) +$$

$$+ \frac{\lambda \Delta T}{2i\omega_q} \left(E \int d\xi R_o + \frac{2i}{\omega_1} \frac{\partial E}{\partial \theta} \int d\xi F_0 U_0 \right) e^{i\omega_1 \Delta T} + c.c.$$

$$U_2 (\theta, \xi; T) = -\Delta T U_0 \frac{\partial u_o}{\partial \theta} - \frac{\lambda \Delta T}{\omega_o^2} \int d\xi \left(U_0^2 \frac{\partial R_o}{\partial \theta} + 2F_0 U_0 \frac{\partial u_o}{\partial \theta} \right) +$$

$$+ \frac{\lambda U_0}{\omega_1^2} \frac{\partial E}{\partial \theta} e^{i\omega_1 \Delta T} - \frac{\lambda^2 \Delta T}{2\omega_1 \omega_q} \left(E \int d\xi R_o + \frac{2i}{\omega_1} \frac{\partial E}{\partial \theta} \int d\xi F_0 U_0 \right) e^{i\omega_1 \Delta T} + \tag{10}$$

$$+\frac{\lambda^2}{2i\omega_q\omega_1^2}\left(E\int d\xi R_o+\frac{2i}{\omega_1}\frac{\partial E}{\partial\theta}\int d\xi F_0U_0\right)e^{i\omega_1\Delta T}+c.c.$$

$$F_2\left(\theta,\xi;T\right)=-\left(U_0\frac{\partial R_o}{\partial\theta}+F_0\frac{\partial u_o}{\partial\theta}\right)\Delta T+\frac{\lambda F_0}{\omega_1^2}\frac{\partial E}{\partial\theta}e^{i\omega_1\Delta T}+c.c.$$

In a way similar to the above we write the third order equations as

$$\frac{\partial F_3}{\partial T}+F_0\frac{\partial U_2}{\partial\theta}+\frac{\partial}{\partial\theta}\left(F_1U_1\right)+U_0\frac{\partial F_2}{\partial\theta}=0,$$

$$\frac{\partial U_3}{\partial T}+U_1\frac{\partial U_1}{\partial\theta}+U_0\frac{\partial U_2}{\partial\theta}=\lambda V_3,$$

$$\frac{\partial^2V_3}{\partial T^2}+2\gamma\frac{\partial V_3}{\partial T}+\omega^2V_3=\int d\xi\left[F_0\frac{\partial U_3}{\partial T}+R_o\frac{\partial U_2}{\partial T}+\frac{\partial\left(F_2U_1\right)}{\partial T}+U_0\frac{\partial F_3}{\partial T}\right].$$

Solving the equation for V_3

$$\frac{\partial^2V_3}{\partial T^2}+2\gamma\frac{\partial V_3}{\partial T}+\omega_o^2V_3=$$

$$=\int d\xi\left[-2F_0U_1\frac{\partial U_1}{\partial\theta}-2F_0U_0\frac{\partial U_2}{\partial\theta}-2U_0\frac{\partial\left(R_oU_1\right)}{\partial\theta}\right]+$$

$$+\int d\xi\left(\lambda R_oV_2+\lambda V_1F_2-U_0^2\frac{\partial F_2}{\partial\theta}\right)$$

that can be obtained by combining the third order equations, and subsequently solving the two other equations for U_3 and F_3 we obtain the third order solution:

$$V_3\left(\theta;T\right)=\frac{\lambda\Delta T}{2i\omega_q}\left\{\frac{2i}{\omega_1}\frac{\partial}{\partial\theta}\left[E\int d\xi\left(u_oF_0+U_0R_o\right)\right]-\right.$$

$$-\frac{3}{\omega_1^2}\left(\int d\xi F_0U_0^2\right)\frac{\partial^2E}{\partial\theta^2}+\frac{1}{2i\omega_q}\left(1-i\omega_q\Delta T\right)\left[\int d\xi\left(U_0\frac{\partial R_o}{\partial\theta}+F_0\frac{\partial u_o}{\partial\theta}\right)\right]E+$$

$$\left.+\frac{\lambda}{4\omega_q^2}\left(1-i\omega_q\Delta T\right)\left(\int d\xi R_o\right)\left[E\int d\xi R_o+\frac{2i}{\omega_1}\left(\int d\xi F_0U_0\right)\frac{\partial E}{\partial\theta}\right]\right\}e^{i\omega_1\Delta T}+$$

$$+c.c.+\text{oscillating terms} \tag{11}$$

$$U_3\left(\theta,\xi;T\right)=-u_o\frac{\partial u_o}{\partial\theta}\Delta T+\frac{\lambda^2}{2\gamma\omega_o^2}\frac{\partial\left|E\right|^2}{\partial\theta}e^{-2\gamma\Delta T}+\text{oscillating terms}$$

$$F_3\left(\theta,\xi;T\right)=-\frac{\partial\left(R_ou_o\right)}{\partial\theta}\Delta T+\text{oscillating terms}$$

Next we collect the secular terms that would contribute to the amplitude equations when applying the RG procedure. Setting now $\varepsilon=1$ we write down the part of the solution of the system (7) that has to be renormalized

$$F_{RG}\left(\theta,\xi;T,T_0\right)=\widetilde{F}\left(\theta,\xi;T_0\right)-\Delta T\frac{\partial}{\partial\theta}\left[\widetilde{F}\left(\theta,\xi;T_0\right)\widetilde{U}\left(\theta,\xi;T_0\right)\right],$$

$$U_{RG}(\theta,\xi;T,T_0) = \widetilde{U}(\theta,\xi;T_0) - \Delta T \widetilde{U}(\theta,\xi;T_0) \frac{\partial}{\partial \theta} \widetilde{U}(\theta,\xi;T_0) +$$

$$+ \frac{\lambda^2}{2\gamma\omega_o^2} \frac{\partial |E(\theta;T_0)|^2}{\partial \theta} e^{-2\gamma\Delta T},$$

$$V_{RG}(\theta;T,T_0) = \left\{ E + \frac{\lambda \Delta T}{2i\omega_q} \left[E \int d\xi R_o + \frac{2i}{\omega_1} \frac{\partial}{\partial \theta} \left(E \int d\xi \widetilde{F}\widetilde{U} \right) - \right. \right. \tag{12}$$

$$- \frac{3}{\omega_1^2} \left(\int d\xi F_0 U_0^2 \right) \frac{\partial^2 E}{\partial \theta^2} - \frac{iE}{2\omega_q} \frac{\partial}{\partial \theta} \left(\int d\xi \widetilde{F}\widetilde{U} \right) +$$

$$\left. \left. + \frac{\lambda}{4\omega_q^2} \left(\int d\xi R_o \right) \left(E \int d\xi R_o + \frac{2i}{\omega_1} \frac{\partial E}{\partial \theta} \int d\xi F_0 U_0 \right) \right] \right\} e^{i\omega_1 \Delta T} + c.c.$$

where

$$\widetilde{F}(\theta,\xi;T_0) = F_0(\xi) + R_o(\theta,\xi;T_0) \quad ; \quad \widetilde{U}(\theta,\xi;T_0) = U_0(\xi) + u_o(\theta,\xi;T_0).$$

Following Kunihiro [4] we represent the solution (12) as a family of trajectories or curves $\{\Re_{T_0}\} = [F_{RG}(T_0), U_{RG}(T_0), V_{RG}(T_0)]$, being parameterized with T_0. The RG equations are defined as the envelope equations for the one-parameter family $\{\Re_{T_0}\}$:

$$\left. \left(\frac{\partial F_{RG}}{\partial T_0}, \frac{\partial U_{RG}}{\partial T_0}, \frac{\partial V_{RG}}{\partial T_0} \right) \right|_{T_0=T} = 0. \tag{13}$$

From the above definition (13) it is straightforward to obtain the desired RG equations:

$$\frac{\partial \widetilde{F}}{\partial T} + \frac{\partial}{\partial \theta} \left(\widetilde{F}\widetilde{U} \right) = 0,$$

$$\frac{\partial \widetilde{U}}{\partial T} + \widetilde{U} \frac{\partial \widetilde{U}}{\partial \theta} = - \frac{\lambda^2}{\omega_o^2} \frac{\partial \left| \widetilde{E} \right|^2}{\partial \theta},$$

$$\frac{2i\omega_q}{\lambda} \left(\frac{\partial}{\partial T} + \frac{\partial}{\partial \theta} + \gamma \right) \widetilde{E} = \widetilde{E} \int d\xi \left(\widetilde{F} - F_0 \right) + \frac{2i}{\omega_1} \frac{\partial}{\partial \theta} \left(\widetilde{E} \int d\xi \widetilde{F}\widetilde{U} \right) - \tag{14}$$

$$- \frac{3}{\omega_1^2} \left(\int d\xi F_0 U_0^2 \right) \frac{\partial^2 \widetilde{E}}{\partial \theta^2} - \frac{i}{2\omega_q} \widetilde{E} \frac{\partial}{\partial \theta} \left(\int d\xi \widetilde{F}\widetilde{U} \right) +$$

$$+ \frac{\lambda}{4\omega_q^2} \left[\int d\xi \left(\widetilde{F} - F_0 \right) \right] \left[\widetilde{E} \int d\xi \left(\widetilde{F} - F_0 \right) + \frac{2i}{\omega_1} \frac{\partial}{\partial \theta} \left(\widetilde{E} \int d\xi \widetilde{F}\widetilde{U} \right) \right],$$

where

$$\widetilde{E}(\theta;T) = E(\theta;T) e^{-\gamma T}.$$

The final step consists in defining the envelope distribution function $G(\theta,v;T)$ by the Radon transform

$$G(\theta,v;T) = \int d\xi \widetilde{F}(\theta,\xi;T) \delta \left[v - \widetilde{U}(\theta,\xi;T) \right]. \tag{15}$$

By virtue of (15) the system of RG equations (14) is equivalent to the following system of equations for the envelope distribution function $G(\theta, v; T)$ and the resonator voltage amplitude $\widetilde{E}(\theta; T)$:

$$\frac{\partial G}{\partial T} + v\frac{\partial G}{\partial \theta} - \frac{\lambda^2}{\omega_o^2}\frac{\partial \left|\widetilde{E}\right|^2}{\partial \theta}\frac{\partial G}{\partial v} = 0, \tag{16}$$

$$\frac{2i\omega_q}{\lambda}\left(\frac{\partial}{\partial T} + \frac{\partial}{\partial \theta} + \gamma\right)\widetilde{E} = \widetilde{E}\int dv\,(G - f_0) + \frac{2i}{\omega_1}\frac{\partial}{\partial \theta}\left(\widetilde{E}\int dvvG\right) -$$

$$-\frac{3}{\omega_1^2}\left[\int dvv^2 f_0(v)\right]\frac{\partial^2 \widetilde{E}}{\partial \theta^2} - \frac{i}{2\omega_q}\widetilde{E}\frac{\partial}{\partial \theta}\left(\int dvvG\right) +$$

$$+\frac{\lambda}{4\omega_q^2}\left[\int dv\,(G - f_0)\right]\left[\widetilde{E}\int dv\,(G - f_0) + \frac{2i}{\omega_1}\frac{\partial}{\partial \theta}\left(\widetilde{E}\int dvvG\right)\right]. \tag{17}$$

The system of equations (16) and (17) provides a complete description of nonlinear particle-wave interaction. It governs slow processes of beam pattern dynamics through the evolution of the amplitude functions. In (16) one can immediately recognize the Vlasov equation for the envelope distribution function $G(\theta, v; T)$ with the ponderomotive force, due to fast oscillations at frequency close to the resonant frequency. It may be worth noting that the system (16) and (17) intrinsically contains the nonlinear Landau damping mechanism, a fact that will become apparent from the treatment in the next Section.

III. DERIVATION OF THE GENERALIZED GINZBURG-LANDAU EQUATION.

In order to solve equation (16) we perform a Fourier transform and obtain:

$$(\Omega - kv)\,G(\chi) =$$

$$-\frac{\lambda^2}{(2\pi)^4\omega_o^2}\int d\chi_1 d\chi_2 d\chi_3 \delta(\chi - \chi_1 - \chi_2 - \chi_3)(k - k_1)\frac{\partial G(\chi_1)}{\partial v}\widetilde{E}(\chi_2)\widetilde{E}^*(\chi_3), \tag{18}$$

where

$$\chi = (k, \Omega) \qquad ; \qquad \delta(\chi) = \delta(k)\,\delta(\Omega)$$

and the Fourier transform of a generic function $g(\theta; T)$ is defined as

$$g(\theta; T) = \frac{1}{(2\pi)^2}\int d\Omega dk g(k; \Omega)\,e^{i(k\theta - \Omega T)},$$

$$g(k; \Omega) = \int d\theta dT g(\theta; T)\,e^{-i(k\theta - \Omega T)},$$

$$g^*(k; \Omega) = [g(-k; -\Omega)]^*.$$

Solving equation (18) perturbatively we represent its solution in the form:

$$G(\chi) = (2\pi)^2 f_0(v)\,\delta(\chi) + \widetilde{G}(\chi) \quad ; \quad \widetilde{G}(\chi) = \sum_{n=1}^{\infty} G_n(\chi), \tag{19}$$

where

$$G_1(\chi) = -\frac{\lambda^2}{(2\pi)^2 \omega_o^2} \frac{k}{\Omega - kv} \frac{\partial f_0}{\partial v} \int d\chi_1 d\chi_2 \delta(\chi - \chi_1 - \chi_2) \widetilde{E}(\chi_1) \widetilde{E}^*(\chi_2), \tag{20}$$

$$G_n(\chi) = -\frac{\lambda^2}{(2\pi)^4 \omega_o^2} \frac{1}{\Omega - kv} *$$

$$* \int d\chi_1 d\chi_2 d\chi_3 \delta(\chi - \chi_1 - \chi_2 - \chi_3)(k - k_1) \frac{\partial G_{n-1}(\chi_1)}{\partial v} \widetilde{E}(\chi_2) \widetilde{E}^*(\chi_3). \tag{21}$$

The Fourier transform of equation (17) yields the linear dispersion relation

$$\Omega = -i\gamma + k - \frac{\lambda \langle v \rangle_0}{\omega_q \omega_1} k + \frac{3\lambda \langle v^2 \rangle_0}{2\omega_q \omega_1} k^2 =$$

$$= k - \frac{\lambda \langle v \rangle_0}{\omega_o^2} k + \frac{3\lambda \langle v^2 \rangle_0}{2\omega_q \omega_o^2} k^2 - i\gamma \left(1 - \frac{\lambda \langle v \rangle_0}{\omega_q \omega_o^2} k + \frac{3\lambda \langle v^2 \rangle_0}{\omega_o^4} k^2\right). \tag{22}$$

The integrals over v of the envelope distribution function $G(\theta, v; T)$

$$I_0(\theta; T) = \int dv G(\theta, v; T) = \frac{1}{(2\pi)^2} \int dv d\Omega dk G(k, v; \Omega) e^{i(k\theta - \Omega T)},$$

$$I_1(\theta; T) = \int dv v G(\theta, v; T) = \frac{1}{(2\pi)^2} \int dv d\Omega dk v G(k, v; \Omega) e^{i(k\theta - \Omega T)}$$

entering equation (17) can be computed in a straightforward manner. Substituting the solution (19)-(21) into the above equations with the linear dispersion relation (22) in hand, up to second order in $G(k, v; \Omega)$, we find

$$I_0(\theta; T) = 1 - W\left[\left|\widetilde{E}(\theta; T)\right|\right] + ..., \tag{23}$$

$$I_1(\theta; T) = 1 - \left(1 - \frac{\lambda}{\omega_o^2}\right) W\left[\left|\widetilde{E}(\theta; T)\right|\right] + ..., \tag{24}$$

where the function W is defined as

$$W(z) = \frac{\lambda^2}{\omega_o^2 \sigma_v^2}(1 + i\gamma_L)\left(1 - \frac{\lambda^2}{2\omega_o^2 \sigma_v^2} z^2\right) z^2, \tag{25}$$

and

$$\gamma_L = \frac{\lambda}{\omega_o^2 \sigma_v} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\lambda^2}{2\omega_o^4 \sigma_v^2}\right) \tag{26}$$

is the Landau damping factor. In the above calculations the equilibrium distribution function has been taken to be the Gaussian one

$$f_0(v) = \frac{1}{\sigma_v \sqrt{2\pi}} \exp\left[-\frac{(v-1)^2}{2\sigma_v^2}\right],$$

where

$$\sigma_v = \frac{|k_o| \sigma_E}{\omega_s}$$

and σ_E is the r.m.s. of the energy error, proportional to the longitudinal beam temperature. By substitution of the expressions (23) and (24) into equation (17) we arrive at the generalized Ginzburg-Landau equation:

$$\frac{2i\omega_q}{\lambda}\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial\theta}+\gamma\right)\widetilde{E}=-\frac{3\left(1+\sigma_v^2\right)}{\omega_1^2}\frac{\partial^2\widetilde{E}}{\partial\theta^2}+\frac{2i}{\omega_1}\frac{\partial\widetilde{E}}{\partial\theta}-\left(1-\frac{\lambda}{4\omega_q^2}W\right)W\widetilde{E}-$$

$$-\frac{2i}{\omega_1}\frac{\lambda}{4\omega_q^2}W\frac{\partial\widetilde{E}}{\partial\theta}+\frac{i}{2\omega_q}\left(1-\frac{\lambda}{\omega_o^2}\right)\widetilde{E}\frac{\partial W}{\partial\theta}-\frac{2i}{\omega_1}\left(1-\frac{\lambda}{\omega_o^2}\right)\left(1-\frac{\lambda}{4\omega_q^2}W\right)\frac{\partial}{\partial\theta}\left(W\widetilde{E}\right).$$

It can be further cast to a simpler form by introducing the rescaled independent and dependent variables according to

$$\tau=\frac{\lambda T}{2\omega_q}\quad;\quad x=\frac{\omega_o}{\sqrt{3\left(1+\sigma_v^2\right)}}\left(\theta-T+\frac{\lambda T}{\omega_o\omega_q}\right)\quad;\quad\Psi=\frac{|\lambda|\,\widetilde{E}}{\omega_o\sigma_v}.$$

Then the generalized Ginzburg-Landau equation for the amplitude of the resonator voltage takes its final form:

$$i\frac{\partial\Psi}{\partial\tau}+ib\gamma\Psi=-\left(1-\frac{2i\gamma}{\omega_o}\right)\frac{\partial^2\Psi}{\partial x^2}+a\gamma\frac{\partial\Psi}{\partial x}-\left(1-b_1W\right)W\Psi-$$

$$-ab_1\left(\gamma+i\omega_o\right)W\frac{\partial\Psi}{\partial x}-a_1\left(\gamma+i\omega_o\right)\left(1-b_1W\right)\frac{\partial}{\partial x}\left(W\Psi\right)+\frac{ia_1b_1b\omega_o^2}{2}\Psi\frac{\partial W}{\partial x},\tag{27}$$

where

$$a=\frac{2}{\omega_o\sqrt{3\left(1+\sigma_v^2\right)}}\qquad;\qquad b=\frac{2\omega_q}{\lambda},$$

$$b_1=\frac{1}{\lambda b^2}\qquad;\qquad a_1=a\left(1-4b_1\right),$$

and the function $W\left(|\Psi|\right)$ is given now by the simple expression:

$$W\left(|\Psi|\right)=\left(1+i\gamma_L\right)\left(1-\frac{1}{2}|\Psi|^2\right)|\Psi|^2.\tag{28}$$

The generalized Ginzburg-Landau equation (27) is known [7] to provide the basic framework for the study of many properties of non equilibrium systems, such as existence and interaction of coherent structures, generic onset of travelling wave disturbance in continuous media, appearance of chaos. Recent experimental and numerical evidence (see e.g. [1], [2] and the references therein) shows that similar behavior is consistent with the propagation of charged particle beams, and the generalized Ginzburg-Landau equation we have derived could represent the appropriate analytical model to study the above mentioned phenomena.

IV. CONCLUDING REMARKS.

As a result of the investigation performed we have shown that a coasting beam under the influence of a resonator impedance exhibits spatial-temporal patterns modulated by an envelope (amplitude) function, which varies slowly compared to the fast time and short wavelength scales of the pattern itself. Extracting long wavelength and slow time scale behavior of the system we have derived a set of coupled nonlinear evolution equations for the beam envelope distribution function and voltage amplitude. We have further shown that the amplitude of the nonlinear wave satisfies a generalized Ginzburg-Landau equation.

It is worthwhile to mention that the analytical framework presented here bears rather general features. It provides complete kinetic description of slow, fully nonlinear particle-wave interaction process, and allows higher order corrections to the generalized Ginzburg-Landau equation to be taken into account.

V. ACKNOWLEDGEMENTS.

The author is indebted to Pat Colestock for many helpful discussions concerning the subject of the present paper, and to David Finley and Steve Holmes for their enlightened support of this work.

---

[1] *S.I. Tzenov and P.L. Colestock*, "Solitary Waves on a Coasting High-Energy Stored Beam", **Fermilab-Pub-98-258**, Fermilab, Batavia, 1998.
physics/9808032, 24 August 1998.

[2] *P.L. Colestock, L.K. Spentzouris and S.I. Tzenov*, "Coherent Nonlinear Phenomena in High Energy Synchrotrons: Observations and Theoretical Models", **International Symposium on Near Beam Physics**, Fermilab, September 22-24, 1997, R.A. Carrigan and N.V. Mokhov eds.,Fermilab, Batavia, 1998, pp 94-104.
physics/9808035, 25 August 1998.

[3] *L.-Y. Chen, N. Goldenfeld and Y. Oono*, "Renormalization Group and Singular Perturbations: Multiple Scales, Boundary Layers and Reductive Perturbation Theory", **Phys. Rev. E,** Vol. **54** (1996) p. 376.

[4] *T. Kunihiro*, "The Renormalization Group Method Applied to Asymptotic Analysis of Vector Fields", **Prog. of Theoretical Physics,** Vol. **97** (1997) p. 179.

[5] *J. Horvath*, **"Topological Vector Spaces and Distributions"**, Addison-Wesley, Reading, Massachusetts, 1966.

[6] *Yu.L. Klimontovich*, **"Statistical Theory of Open Systems"**, Kluwer Academic Publishers, Dordrecht, 1995.

[7] *M.C. Cross and P.C. Hohenberg*, "Pattern Formation Outside of Equilibrium", **Reviews of Modern Physics**, Vol. **65** (1993) p. 851.